

# Resolution structure in HornSAT and CNFSAT

KOBAYASHI, Koji

E-mail:

## Overview

This article describes about the difference of resolution structure and size between HornSAT and CNFSAT.

We can compute HornSAT by using clauses causality. Therefore we can compute proof diagram by using Log space reduction. But we must compute CNFSAT by using clauses correlation. Therefore we cannot compute proof diagram by using Log space reduction, and reduction of CNFSAT is not P-Complete.

## Preparation

In this paper, we use CNF description as follows;

**Definition 1.** About  $F \in CNF$ , we describe the composition of the clauses  $c \in F$  as a subscript. That is,  $c_{i...j...} = (x_i \vee \cdots \overline{x_j} \vee \cdots)$ . The subscript of a capital letter shall be either positive or negative of a variable. For examples,  $c_I, c_{\overline{I}}$  means  $c_I, c_{\overline{I}} \in \{c_i, \overline{c_i}\}, c_I \neq c_{\overline{I}}$

And define resolution of clauses as follows;

**Definition 2.** About resolution, I will use the term “Joint Variable” as variables that positive and negative variable which are included in each antecedents and not included in consequent, and

---

\*To whom correspondence should be addressed

“Positive Antecedent” as antecedent that have positive joint variable, “Negative Antecedent” as antecedent that have negative joint variable. We treat some resolution that have same joint variable. Such case, positive antecedent, negative antecedents and consequents become set of clauses.

## Resolution

We show the character of the resolution.

**Theorem 3.** *In CNF resolution, number of joint variable of each antecedents is one.*

*Proof.* I prove it using reduction to absurdity. We assume that some resolution have 0 or over 2 joint variable.

The case that resolution have 0 joint variable contradicts a condition of the resolution clearly. The case that resolution have 2 joint variable contradicts a condition of the resolution because  $c_{IJp...} \vee c_{\overline{IJ}q...} \rightarrow c_{Jp...}\overline{J}q... = \top$ .

Therefore, this theorem was shown than reduction to absurdity. □

I introduce topology of deduction system to formula. For simplification, I treat topology as formula.

**Definition 4.** About  $F \in CNF$ , I will use the term “DCNF(Deduction CNF)” as formula that variables value are presence of restrictions of CNF formula clauses. Especially, I will use the term “RCNF(Resolution CNF)” and “ $RCNF(F)$ ” as DCNF that deduction system is resolution principle. Clauses become variables and resolution become clauses in  $RCNF(F)$ . Antecedent become negative variables and consequent become positive variables. And furthermore, RCNF does not include variable that correspond to empty clause.

That is, if

$$F \supset c_{ip...} \wedge c_{iq...},$$

then

$$RCNF(F) \supset (c_{ip...}) \wedge (\overline{c_{iq...}}) \wedge (\overline{c_{ip...}} \vee \overline{c_{iq...}} \vee c_{p...q...}).$$

And RCNF does not include variable correspond to empty clause, therefore sufficiency of  $F$  accords with  $RCNF(F)$ . Resolution consequent is 1 or less, therefore  $RCNF(F) \in HornCNF$ . That is, if  $RCNF(g) = f$ ;

$$RCNF = HornCNF \ni f : \{g \mid g \in CNF\} \xrightarrow{Resolution} \{\top, \perp\}$$

## HornSAT and RCNF

Think  $RCNF(HornCNF)$  complexity. Relation of  $HornCNF$  clauses are causality and we can compute them by using unit resolution. Therefore, we can reduce  $HornCNF$  to  $RCNF(HornCNF)$  by using log space reduction. And  $RCNF \subset HornCNF$ , then  $RCNF$  is P-Complete.

**Theorem 5.**  $f \in P - Complete \mid RCNF \ni f : \{g \mid g \in HornCNF\} \xrightarrow{Resolution} \{\top, \perp\}$

*Proof.* Clearly  $RCNF \subset HornCNF$  and  $RCNF \in P$ , I should show that  $\exists h \in L(h : g \mapsto f)$  (L:Log space reduction). We treat  $h$  as 2-step procedures to simplify this. First,

First, I reduce  $HornCNF$  to at most 3 variables clauses  $HornCNF$ . We can reduce by using same way to reduce  $CNF$  to  $3CNF$ . That is, each clauses change follows with new variables.

$$g \ni c_{I\bar{j}kl} \dots \rightarrow c_{I\bar{j}0} \wedge c_{0\bar{k}1} \wedge c_{1\bar{l}2} \wedge \dots \in g'$$

We can execute this reduction with logarithm space, pointer to consequent, pointer to variable, counter that show already used variables.

Second, I reduce  $c' \in g'$  to  $RCNF(c')$ . We can reduce by adding resolution formula for each clauses. We can reduce HornCNF with unit resolution, therefore it is enough to keep SAT by using resolution formula that variables of antecedent decreases. That is;

$$c_R \rightarrow (x_R) \wedge (\overline{x_R} \vee \overline{x_R})$$

$$c_{P\bar{q}} \rightarrow (x_{P\bar{q}}) \wedge (x_P \vee \overline{x_{P\bar{q}}} \vee \overline{x_q}) \wedge (\overline{x_P} \vee \overline{x_{\bar{P}}})$$

$$c_{I\bar{j}k} \rightarrow (x_{I\bar{j}k}) \wedge (x_{I\bar{k}} \vee \overline{x_{I\bar{j}k}} \vee \overline{x_j}) \wedge (x_{I\bar{j}} \vee \overline{x_{I\bar{j}k}} \vee \overline{x_k}) \wedge (x_I \vee \overline{x_{I\bar{j}}} \vee \overline{x_j}) \wedge (x_I \vee \overline{x_{I\bar{k}}} \vee \overline{x_k}) \wedge (\overline{x_I} \vee \overline{x_{\bar{I}}})$$

We can execute this reduction with logarithm space, pointer to consequent, pointer to variable.

Above two reduction, we can reduce  $HornCNF$  to  $RCNF$ . Both reductions use only logarithm space, we can execute all reduction  $h : g \mapsto g' \mapsto f$  in logarithm space.

Therefor, RCNF is P-Complete. □

## CNFSAT and RCNF

Think  $RCNF(CNF)$  complexity. Relation of  $CNF$  clauses are correlation and we cannot compute them by using unit resolution. Therefore, we cannot reduce  $CNF$  to  $RCNF(CNF)$  by using log space reduction. And  $RCNF \subset HornCNF$ . That is,  $RCNF$  is not P-Complete.

Afterward, we show some  $CNF$  that  $RCNF$  is not P-Complete. First, we think the formula that each reduction depend whole formula.

**Definition 6.** We will use the term “S3CNF(3-Simplex CNF)” to;

$$t_{PQR} = c_{\overline{PQ}} \wedge c_{\overline{QR}} \wedge c_{\overline{PR}} \wedge c_{PQR}$$

and “S4CNF(4-Simplex CNF)” to;

$$T_{PQR} = c_{\overline{PQR}} \wedge c_{P\overline{QR}} \wedge c_{\overline{P}Q\overline{R}} \wedge c_{PQR}$$

and “SCNF” to  $S3CNF \cup S4CNF$ .

Second, we think the formula that consist of  $SCNF$ .

**Definition 7.**  $f \in CNF$  that consist of  $SCNF$ , we will use term “CCNF(Chaotic CNF)” if  $f$  satisfy follow condition.

The Graph that each  $SCNF \ni t \subset f$  are nodes and each variables are edges.

a) This Graph is 3-Moore Graph.

b) When this graph girth is  $2k + 1$ , all circuit include  $S4CNF$  that number is  $k \times c_0 \mid c_0 : const(c_0 > 1)$ .

Next, we think that  $RCNF(CCNF)$  is not P-Complete. We show that  $RCNF(CCNF)$  is not polynomial size and we cannot treat  $RCNF(CCNF)$  by using log space reduction.

**Theorem 8.**  $f \in CCNF$  are exists that  $RCNF(CCNF)$  is not polynomial size.

*Proof.* I prove it using reduction to absurdity. We assume that we can reduce all all  $f \in CCNF$  to  $RCNF(CCNF)$  in polynomial size. From this assumption, number of  $RCNF(CCNF)$  consequent stay in polynomial size.

From  $S4CNF$  structure, each  $S4CNF$  resolution's consequents are include over one joint variables. Therefore, next resolution must include another clause as antecedent. That is,  $S4CNF$  resolution become product of positive antecedents and negative antecedents. And  $f$  is Moore Graph structure, therefore it is necessary over girth  $2k + 1$  clauses to appear same clause in processing resolution antecedent. Resolution that one of antecedent is  $S4CNF$  have consequents size twice of antecedents size. Therefore, consequents size become  $2^{k \times c_0}$ . On the other side, size of 3-Moore Graph is  $1 + 3 \sum_{i=0}^{k-1} (3-1)^i = 1 + 3 \times (2^k - 1)$ . Therefore, ratio of size of  $f$  and consequents of  $RCNF(f)$  is;

$$O\left(\frac{|f|}{|RCNF(f)|}\right) = O\left(\frac{2^{k \times c_0}}{1 + 3 \times (2^k - 1)}\right) \rightarrow O(c^k) \quad (as \quad k \gg 0)$$

And  $RCNF(f)$  consequents is not in polynomial size and contradicts a condition that  $RCNF(CCNF)$  consequent stay in polynomial size.

Therefore, this theorem was shown than reduction to absurdity.  $\square$

**Theorem 9.**  $(RCNF \ni f : \{g \mid g \in CCNF\} \xrightarrow{Resolution} \{\top, \perp\}) \rightarrow (\forall h \in L(h : g \mapsto f))$

*Proof.* I prove it using reduction to absurdity. We assume that  $h$  exists that all  $g \mapsto f$  satisfy this theorem. Because  $h \in L$ ,  $h$  classify at most polynomial size. Therefore, size of  $f$  (that is target of  $h$ ) also stay polynomial size.

But mentioned above 8,  $CCNF$  have  $f$  that is not in polynomial size. Therefore, there exists  $f$  that is  $L \not\ni h : g \rightarrow f$  and contradicts a condition that  $h$  exist that all  $f$  of  $g \mapsto f$  in polynomial size.

Therefore, this theorem was shown than reduction to absurdity.  $\square$

**Theorem 10.**  $f \notin P-Complete \mid RCNF \ni f : \{g \mid g \in CCNF\} \xrightarrow{Resolution} \{\top, \perp\}$

*Proof.* Mentioned above 9, there is no log space reduction that reduce  $g \in CCNF$  to  $f \in RCNF$ . Therefore,  $f$  is not P-Complete.  $\square$

## References

- (1) Michael Sipser, (translation) OHTA Kazuo, TANAKA Keisuke, ABE Masayuki, UEDA Hiroki, FUJIOKA Atsushi, WATANABE Osamu, Introduction to the Theory of COMPUTATION  
Second Edition, 2008
- (2) HAGIYA Masami, NISHIZAKI Shinya, Mechanism of Logic and Calculation, 2007